Physics 319
Classical Mechanics

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Lecture 10

Solutions Qualitatively

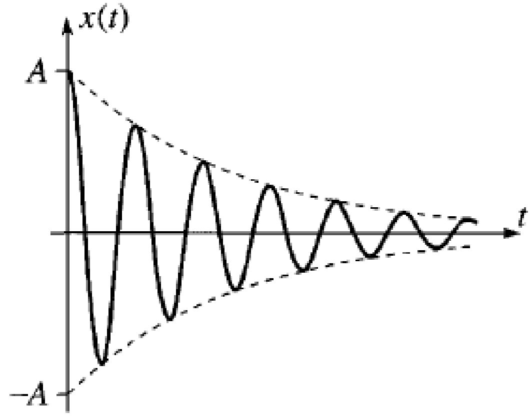


Figure 5.11 Underdamped oscillations can be thought of as simple harmonic oscillations with an exponentially decreasing amplitude $Ae^{-\beta t}$. The dashed curves are the envelopes, $\pm Ae^{-\beta t}$.

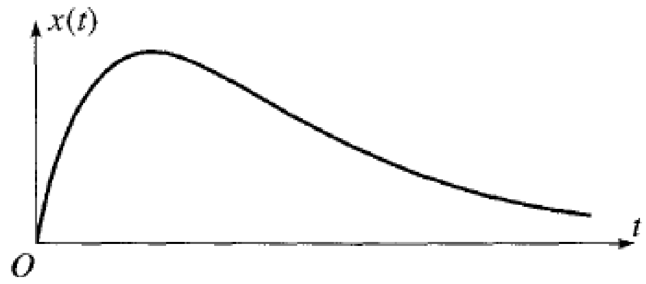


Figure 5.12 Overdamped motion in which the oscillator is kicked from the origin at $t = 0$. It moves out to a maximum displacement and then moves back toward O asymptotically as $t \to \infty$.

Energy Again

- Use work-energy theorem to estimate the energy loss during one cycle in underdamped case

$$f_{frict} = -b\dot{x} \qquad x(t) \doteq A\cos(\omega t - \delta)$$

$$\Delta T = W_{frict} = -\oint b\dot{x}dx = -\int_0^{2\pi/\omega} b\omega A\omega A\left(-\sin(\omega t - \delta)\right)^2 dt$$

$$= -\frac{b}{m}kA^2\frac{\pi}{\omega}$$

- Energy damping rate

$$P_{diss} = -\frac{d}{dt}\left[\frac{kA^2}{2}\right] = -\frac{\Delta T}{\tau_{period}} = \frac{b}{m}kA^2\frac{\pi}{\omega}\frac{\omega}{2\pi} = \frac{b}{m}\frac{kA^2}{2}$$

- Energy damping rate b/m, velocity damping rate $\beta = b/2m$

Driven Damped Motion

- Suppose now a driving force F_{ext} acting on the oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_{ext} / m$$

- Linear Inhomogeneous ODE. General solution: particular solution plus general solution to the inhomogeneous problem.

$$x(t) = x_p(t) + B_1 x_1(t) + B_2 x_2(t)$$

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = F_{ext} / m$$

$$\ddot{x}_{1,2} + 2\beta\dot{x}_{1,2} + \omega_0^2 x_{1,2} = 0$$

- Because linear superposition valid if

$$\ddot{x}_{p,1} + 2\beta\dot{x}_{p,1} + \omega_0^2 x_{p,1} = F_{ext,1} / m$$

$$\ddot{x}_{p,2} + 2\beta\dot{x}_{p,2} + \omega_0^2 x_{p,2} = F_{ext,2} / m,$$

$x_{p,1} + x_{p,2}$ is particular solution for $\left(F_{ext,1} + F_{ext,2}\right) / m$

- General method: solve for sinusoidal driving terms and sum

For Sinusoidal Driving

$$F_{ext} = F_0\left(\cos\omega t + i\sin\omega t\right) = F_0 e^{i\omega t}$$

$$z_p \equiv x + iy$$

$$\ddot{z}_p + 2\beta\dot{z}_p + \omega_0^2 z_p = \frac{F_0}{m}e^{i\omega t}$$

- Particular solution for exponential driving force is just the exponential

$$z_p\left(t\right) = C_+ e^{i\omega t}$$

$$\left(-\omega^2 + i\omega 2\beta + \omega_0^2\right)Ce^{i\omega t} = \frac{F_0}{m}e^{i\omega t}$$

$$C_+ = \frac{F_0}{m}\left(-\omega^2 + i\omega 2\beta + \omega_0^2\right)^{-1} = Ae^{-i\delta}$$

$$A^2 = C_+ C_+^* \qquad \delta = \tan^{-1}\frac{2\omega\beta}{\omega_0^2 - \omega^2}$$

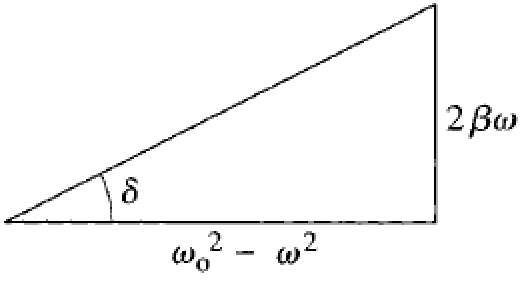


Figure 5.14 The phase angle δ is the angle of this triangle.

General Solution

- The general solution to the forced harmonic oscillator is

$$x(t) = x_p(t) + B_1 x_1(t) + B_2 x_2(t)$$

- But x_1 and x_2 are damped (vanish as $t \rightarrow \infty$). They are transient solutions

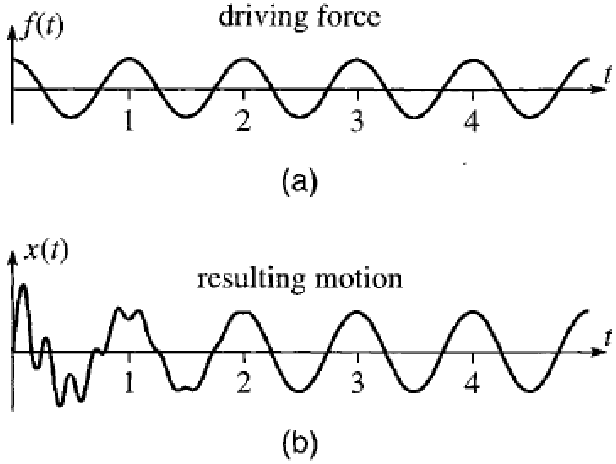


Figure 5.15 The response of a damped, linear oscillator to a sinusoidal driving force, with the time t shown in units of the drive period. **(a)** The driving force is a pure cosine as a function of time. **(b)** The resulting motion for the initial conditons $x_o = v_o = 0$. For

Amplitude

$$A^2 = \frac{F_0^2}{m^2} \frac{1}{4\beta^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}$$

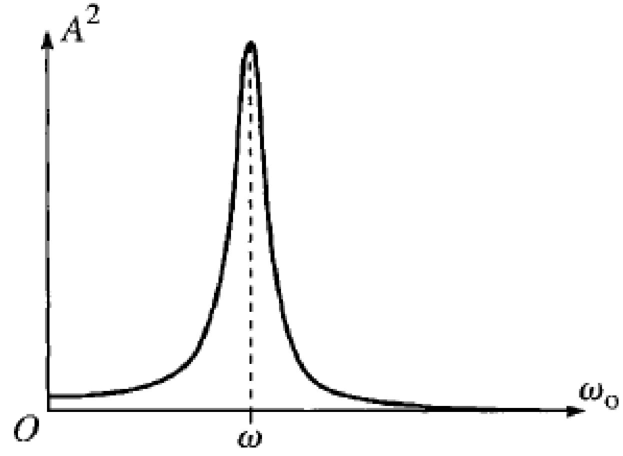


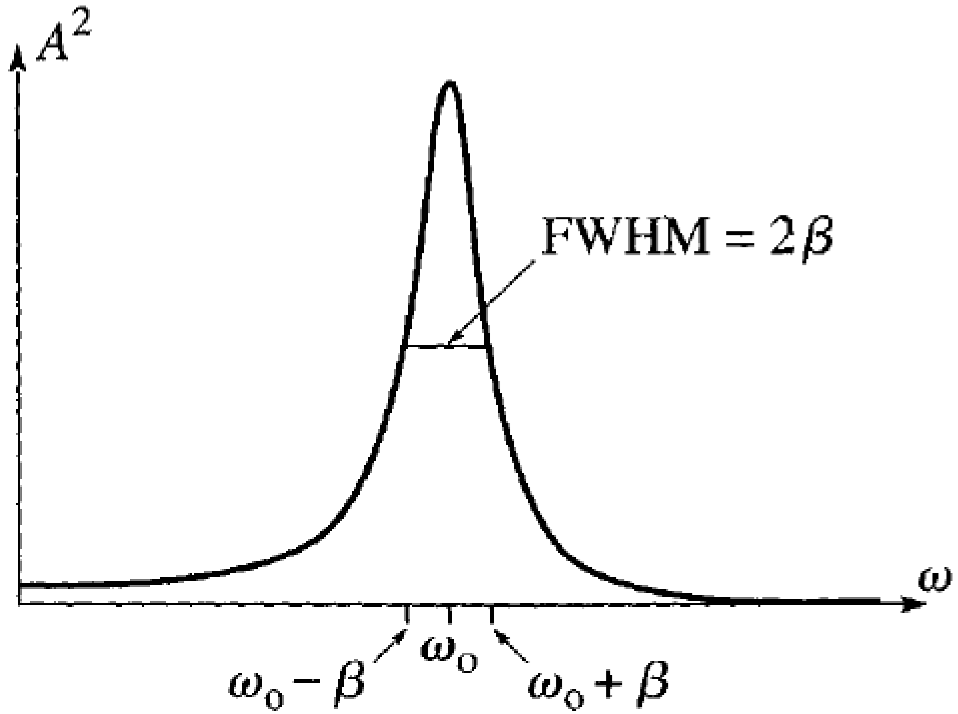
Figure 5.16 The amplitude squared, A^2, of a driven oscillator, shown as a function of the natural frequency ω_o, with the driving frequency ω fixed. The response is dramatically largest when ω_o and ω are close.

- Resonance curve

Resonance Q Value

- The width of the resonance curve is characterized by the Q-value: FWHM resonance curve ω_0/Q

$$\left(\omega^2-\omega_0^2\right)=\pm 2\beta\omega \to \omega \doteq \omega_0 \pm \beta \to Q=\omega_0/2\beta$$



Other Mnemonics for Q

- π times the number of oscillations in a decay time

$$Q = \omega_0 / 2\beta = \pi \frac{1}{\beta} \frac{1}{T_{period}}$$

- Inverse of the fractional energy dissipated in one radian of oscillation

$$Q = 2\pi \frac{\text{Energy Stored}}{\text{Energy Dissipated in one cycle}}$$

$$= \frac{\text{Energy Stored}}{\text{Energy Dissipated in one radian}} = 2\pi \left(\frac{kA^2}{2} \right) / \left(2\beta \frac{kA^2}{2} T_{period} \right)$$

- ω_0 times ratio between stored energy and average power loss

$$Q = \omega_0 \frac{\text{Energy Stored}}{\text{Average Power Loss}}$$

Phase Shift

$$\delta = \tan^{-1}\frac{2\omega\beta}{\omega_0^2 - \omega^2}$$

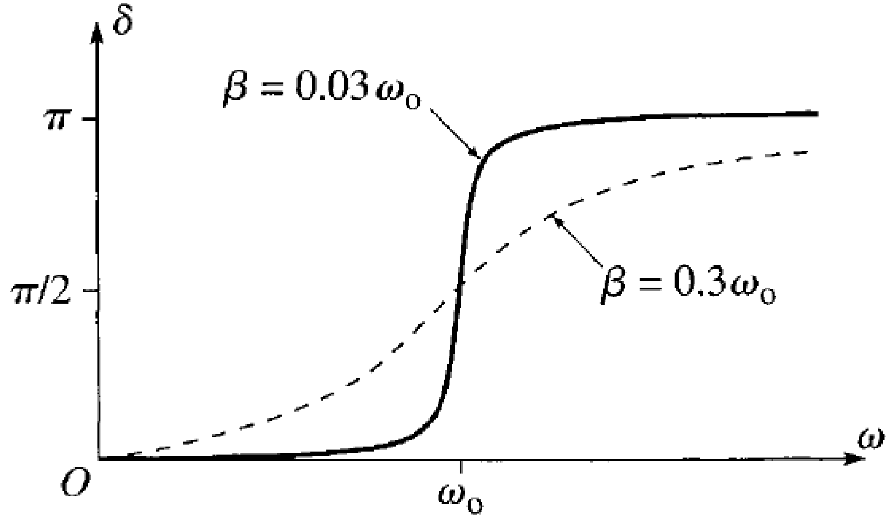


Figure 5.19 The phase shift δ increases from 0 through $\pi/2$ to π as the driving frequency ω passes through resonance. The narrower the resonance, the more suddenly this increase occurs. The solid curve is for a relatively narrow resonance ($\beta = 0.03\omega_0$ or $Q = 16.7$), and the dashed curve is for a wider resonance ($\beta = 0.3\omega_0$ or $Q = 1.67$).

Fourier Series

- Decompose driving term into sinusoids and harmonics and sum to get total response. Can do this for strictly periodic driving forces

$$f(t+\tau) = f(t)$$

- Fourier Series (1st Version)

$$f(t) = \sum_{n=0}^{\infty} A_n \cos(2\pi nt/\tau) + B_n \sin(2\pi nt/\tau)$$

$$= \sum_{n=0}^{\infty} A_n \cos(n\omega t) + B_n \sin(\omega nt)$$

- Orthogonality

$$\frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \cos(m\omega t)\cos(n\omega t)\,dt = \delta_{mn} \qquad \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \sin(m\omega t)\sin(n\omega t)\,dt = \delta_{mn}$$

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \cos(m\omega t)\sin(n\omega t)\,dt = 0$$

Fourier Expansion

- Expand general function and use orthogonality

$$f(t) = \sum_{n=0}^{\infty} A_n \cos(2\pi nt/\tau) + B_n \sin(2\pi nt/\tau)$$

$$\int_{-\tau/2}^{\tau/2} f(t)\cos(m\omega t)\,dt = \int_{-\tau/2}^{\tau/2} \left[\sum_{n=0}^{\infty} A_n \cos(n\omega t) + B_n \sin(n\omega t) \right] \cos(m\omega t)\,dt$$

$$= A_m \tau/2 \rightarrow A_m = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t)\cos(m\omega t)\,dt \qquad m \geq 1$$

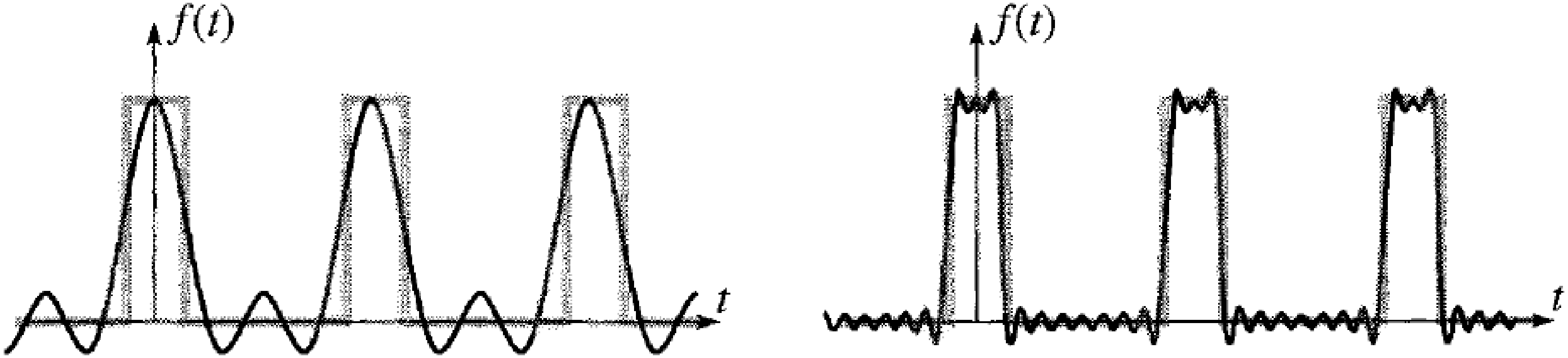
- Similarly

$$B_m = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t)\sin(m\omega t)\,dt \qquad m \geq 1$$

$$A_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t)\,dt$$

Rectangular Pulse

$$A_0 = f_{\max}\frac{\Delta\tau}{\tau} \qquad A_n = \frac{2f_{\max}}{\pi n}\sin\left(\pi n\Delta\tau/\tau\right) \qquad B_n = 0$$



(a) 3 terms (b) 11 terms

Figure 5.23 **(a)** The sum of the first three terms of the Fourier series for the rectangular pulse of Figure 5.22. **(b)** The sum of the first 11 terms.

Oscillator Driven by Periodic Pulse

- Problem

$$\ddot{x} + \left(\omega_0 / Q\right)\dot{x} + \omega_0^2 x = \sum_{n=0}^{\infty} A_n \cos\left(n\omega t\right)$$

- Particular Solution

$$x_{p,n}\left(t\right) = \frac{A_n}{\sqrt{\left(\omega_0^2 - n^2\omega^2\right)^2 + 4\beta^2 n^2 \omega^2}} \cos\left(n\omega t - \delta_n\right)$$

$$\delta_n = \tan^{-1} \frac{2\beta n \omega}{\omega_0^2 - n^2\omega^2}$$

- Because of resonant denominator, in general only a few resonance terms are significant.

Parseval's Theorem (Fourier Series)

- General result in infinite dimensional vector spaces. For case of Fourier series use orthogonality

$$\langle x^2 \rangle = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \left[\sum_{m=0}^{\infty} A_m \cos(m\omega t) + B_m \sin(m\omega t) \right] \times \left[\sum_{n=0}^{\infty} A_n \cos(n\omega t) + B_n \sin(n\omega t) \right] dt$$

$$= A_0^2 + \sum_{n=0}^{\infty} \frac{A_n^2}{2} + \sum_{n=0}^{\infty} \frac{B_n^2}{2}$$

- Very useful also when doing Fourier Transforms

Fourier Series

- The more common definition for Fourier Series is

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t}$$

$$A_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t) e^{-in\omega t} dt \quad \text{complex}$$

- The orthogonality conditions are now

$$\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} e^{im\omega t} e^{-in\omega t} dt = \delta_{mn}$$

and don't have to treat $n = 0$ separately.

- Parseval's Theorem for a real function is

$$\langle x^2 \rangle = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} e^{im\omega t} e^{-in\omega t} dt = A_0^2 + 2\sum_{n=1}^{\infty} A_n A_{-n} = A_0^2 + 2\sum_{n=1}^{\infty} |A_n|^2$$